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Numerical Methods:

Second Semester

Stage: Third

Civil Engineering Department

Syllabus

- ❖Numerical Solutions of Ordinary D.E
- ❖Matrices and their applications
- ❖Interpolation and Curve Fitting
- ❖Numerical Integration
- ❖Boundary Value Problems and Finite Differences
- ❖Numerical Solution for Partial D.E

Numerical Solutions of Ordinary D.E.

If $\dfrac{dy}{dx} = f(x, y)$ *can't be solved by direct integration. The numerical solution techiques are used. To describe various numerical methods for the solution of ordinary D.E., we consider the general first order D.E.*

$$\frac{dy}{dx} = f(x, y), \text{ with initial condition } y(x_o) = y_o$$

and illustrate the theory with respect to this equation. The methods so developed can, in general, be applied to the solution of systems of first order equation and will yield the solution in of the two forms;

$1-A\,series\ for\ y\ in\ terms\ of\ powers\ of\ x,$
$from\ which\ the\ value\ of\ y\ can\ obtained\ by$
$dircect\ substitution.$

$2-\ A\ set\ of\ tabulated\ values\ of\ x\ and\ y.$

$Euler's\ \ Method:$

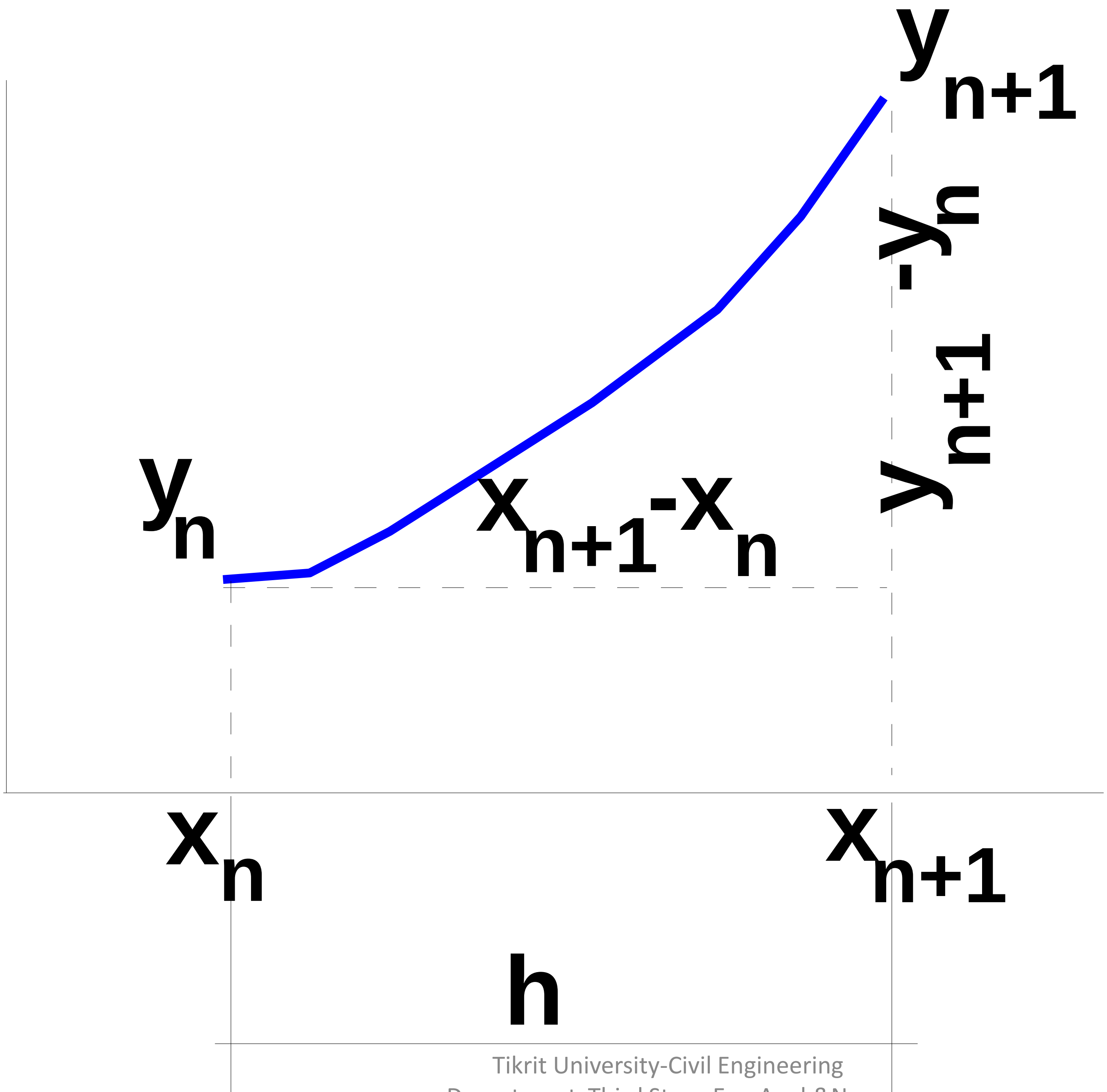
$$Convert\ \frac{dy}{dx}\ to\ \frac{\Delta y}{\Delta x}$$

$$\frac{dy}{dx} = f(x,y) = f(x_n, y_n)$$

$$f(x_n, y_n) = \frac{y_{n+1} - y_n}{x_{n+1} - x_n} = \frac{y_{n+1} - y_n}{h}$$

$$y_{n+1} - y_n = h * f(x_n, y_n)$$

$$\therefore y_{n+1} = y_n + h\, f(x_n, y_n)$$



y_{n+1}
$y_{n+1}-y_n$
y_n
$x_{n+1}-x_n$
x_n
x_{n+1}
h

$$Ex: Consider\ D.E.\ \frac{dy}{dx} = x + y\ with\ condition$$

$$y(0) = 1.\ Find\ \ y\ when\ x = 2\ (let\ h = 0.5)$$

Solution

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$y_1 = y_o + h\, f(x_o, y_o)$$

$$y_1 = 1 + 0.5(0 + 1) = 1.5$$

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$y_2 = 1.5 + 0.5(0.5 + 1.5) = 2.5$$

$$y_3 = 2.5 + 0.5(1.0 + 2.5) = 4.25$$

$$y_4 = 4.25 + 0.5(1.5 + 4.25) = 7.125$$

$$\therefore\ y(2) = 7.125$$

$$Ex:\ D.E.,\frac{dy}{dx}=-y\ with\ y(0)=1.$$

$$Find\ \ y\ when\ x=0.04\ (let\ h=0.01)$$

Solution

$$y_{n+1}=y_n+h\,f(x_n,y_n)$$

$$y_1=y(0.01)=1+0.01(-1)=0.99$$

$$y_2=y(0.02)=0.99+0.01(-0.99)=0.9801$$

$$y_3=y(0.03)=0.9801+0.01(-0.9801)=0.9703$$

$$y_4=y(0.04)=0.9703+0.01(-0.9703)=0.9606$$

$$\therefore\ y(0.04)=0.9606$$

Modified Euler's Method: or

Predictor Corrector Method

This is the simplest of an entire series of method called predictor corrector methods.

1 − Predictor step:

$$\bar{y}_{n+1} = y_n + \Delta x\, f(x_n, y_n) \quad \text{..... from Euler's method}$$

having obtained $\bar{y}_{n+1}$, we now calculate $f(x_n, y_n)$ which is the derivative at the new point $(x_{n+1}, \bar{y}_{n+1})$ and average with the previous derivative $f(x_n, y_n)$ to find the average derivative;

$$\frac{1}{2}[f(x_n, y_n) + f(x_{n+1}, \bar{y}_{n+1})]$$

$2-Now,\ we\ substitute\ this\ average\ value\ into$
$the\ original\ iteration\ equation\ instead\ of$
$f(x_n, y_n)\ to\ get;$

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, \bar{y}_{n+1})]$$

$$Ex:\ D.E.,\frac{dy}{dx}=y,\ y(0)=1.Find\ \ y\ at\ x=1$$

Solution

$$1-Use\ h=1,\ n=1$$

$$\bar{y}_1=y_o+h\,f(x_o,y_o)=1+1(1)=2$$

$$y_1=y_o+\frac{h}{2}[f(x_o,y_o)+f(x_1+\bar{y}_1)]$$

$$=1+\frac{1}{2}[1+2]=2.5$$

$$2 - Use\ h = 0.5,\ n = 2$$

$$\bar{y}_1 = y_o + h\,f(x_o, y_o) = 1 + 0.5(1) = 1.5$$

$$y_1 = y_o + \frac{h}{2}[f(x_o, y_o) + f(x_1 + \bar{y}_1)]$$

$$= 1 + \frac{0.5}{2}[1 + 1.5] = 1.625$$

$$\bar{y}_2 = y_1 + h\,f(x_1, y_1) = 1.625 + 0.5(1.625) = 2.4375$$

$$y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2 + \bar{y}_2)]$$

$$= 1.625 + \frac{0.5}{2}[1.625 + 2.4375] = 2.6406$$

Modified Euler's Method:

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, \bar{y}_{n+1})]$$

$$\bar{y}_{n+1} = y_n + h\, f(x_n, y_n)$$

$$k_1 = h\, f(x_n, y_n)$$

$$k_2 = h\, f(x_{n+1}, y_n + k_1)$$

$$\therefore y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$Ex: D.E., \frac{dy}{dx} = y, y(0) = 1. Find\ y\ at\ x = 1$$

$use\ h = 0.5$

$Solution$

$x_o = 0, \qquad y_o = 1, \qquad h = 0.5$

$k_1 = 0.5(1) = 0.5$

$k_2 = 0.5(1.5) = 0.75$

$$y_1 = 1 + \frac{1}{2}(0.5 + 0.75) = 1.625$$

$x_1 = 0.5, y_1 = 1.625, h = 0.5$

$k_1 = 0.5(1.625) = 0.8125$

$k_2 = 0.5(1.625 + 0.8125) = 1.21875$

$$y_2 = 1.625 + \frac{1}{2}(0.8125 + 1.21875) = 2.640625$$

$Ex: Determine\ the\ value\ of\ y\ when\ x = 0.1$

$given\ that\ y(0) = 1\ and\ y' = x^2 + y$

$Solution$

$1-\ h = 0.05$

$$\bar{y}_1 = y_o + hf(x_o, y_o) = 1 + 0.05(0^2 + 1) = 1.05$$

$$y_1 = y_o + \frac{h}{2}[f(x_o, y_o) + f(x_1, \bar{y}_1)]$$

$$= 1 + \frac{0.05}{2}[1 + \{(0.05)^2 + 1.05\}] = 1.0513$$

$$\bar{y}_2 = y_1 + hf(x_1, y_1)$$

$$= 1.0513 + 0.05(0.05^2 + 1.0513) = 1.10399$$

$$y_2 = 1.0513 + \frac{0.05}{2}[(0.05)^2 + 1.0513 + (0.1)^2 + 1.10399] = 1.10549$$

$$2-\quad x_o=0,\ \ y_o=1,\ \ h=0.05$$

$$k_1=h\,f(x_n,y_n)=0.05(x_o^2+y_o)=0.05+(0^2+1)=0.05$$

$$k_2=h\,f(x_{n+1},y_n+k_1)=0.05[(0+0.5)^2+(1+0.05)]=0.052625$$

$$y_1=y_o+\frac{1}{2}(k_1+k_2)=1+\frac{1}{2}(0.05+0.052625)=1.0513$$

$$x_1=0.05,\ \ y_1=1.0513,\ \ h=0.05$$

$$k_1=0.05[(0.05)^2+1.0513]=0.05269$$

$$k_2=0.05[(0.1)^2+(1.0513+0.05269)]=0.0557$$

$$y_2=1.0513+\frac{1}{2}(0.05269+0.0557)=1.10549$$

$Runge - Kutta\ Method$:

$The\ formula\ for\ (4^{th})\ order\ Runge\ Kutta$

$is\ given\ as\ follows$:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_i, y_i)$$

$$k_2 = hf(x_i + \frac{h}{2}, y_i + \frac{k_1}{2})$$

$$k_3 = hf(x_i + \frac{h}{2}, y_i + \frac{k_2}{2})$$

$$k_4 = hf(x_i + h, y_i + k_3)$$

$$Ex: \frac{dy}{dx} = y,\ y(0) = 1\ \ find\ \ y\ \ at\ \ x = 1,\ h = 1$$

$$Solution$$

$$k_1 = h\ f(x_i, y_i) = 1(1) = 1$$

$$k_2 = h\ f(x_i + \frac{h}{2}, y_n + \frac{k_1}{2}) = 1(1 + \frac{1}{2}) = 1.5$$

$$k_3 = h\ f(x_i + \frac{h}{2}, y_n + \frac{k_2}{2}) = 1(1 + \frac{1.5}{2}) = 1.75$$

$$k_4 = h\ f(x_i + h, y_n + k_3) = 1(1 + 1.75) = 2.75$$

$$y_1 = y_o + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(1 + 2(1.5) + 2(1.75) + 2.75) = 2.70833$$

$$Ex: \frac{dy}{dx} = x^2 + y,\ y(0) = 1\ \ find\ \ y\ at\ x = 1,\ h = 0.5$$

Solution

$$x = 0,\ \ y = 1,\ \ \ h = 0.5$$

$$k_1 = 0.5(0^2 + 1) = 0.5$$

$$k_2 = 0.5[(0 + \frac{0.5}{2})^2 + (1 + \frac{0.5}{2})] = 0.65625$$

$$k_3 = 0.5[(0 + \frac{0.5}{2})^2 + (1 + \frac{0.62625}{2})] = 0.6953$$

$$k_4 = 0.5[(0 + 0.5)^2 + (1 + 0.6953)] = 0.97265$$

$$y_1 = 1 + \frac{1}{6}(0.5 + 2(0.65625 + 0.6953) + 0.97265) = 1.696$$

$$x = 0.5, \quad y = 1.696, \quad h = 0.5$$

$$k_1 = 0.5(0.5^2 + 1.696) = 0.973$$

$$k_2 = 0.5[(0.5 + \frac{0.5}{2})^2 + (1.696 + \frac{0.973}{2})] = 1.3725$$

$$k_3 = 0.5[(0.5 + \frac{0.5}{2})^2 + (1.696 + \frac{1.3725}{2})] = 1.4724$$

$$k_4 = 0.5[(0.5 + 0.5)^2 + (1.696 + 1.4724)] = 2.0842$$

$$y_2 = 1.696 + \frac{1}{6}(0.973 + 2(1.3725 + 1.4724) + 2.0842) = 3.1538$$

$Numerical\ Methods\ in\ General$

$Taylor\ Series$

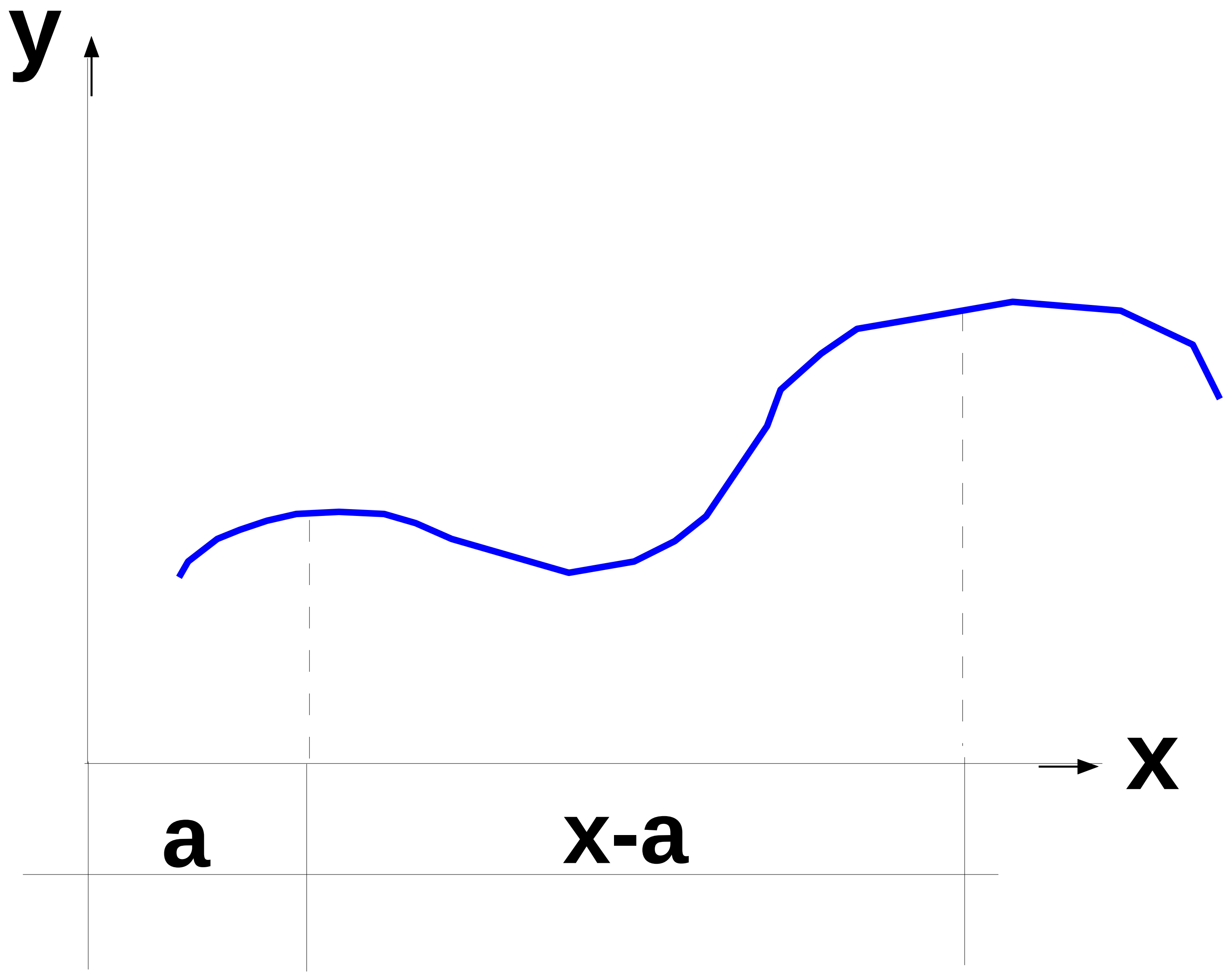
$Let\ (f(x))\ is\ a\ series\ written\ as$

$$f(x) = a_o + a_1(x-a) + a_2(x-a)^2 \ldots\ldots + a_n(x-a)^n$$

$where;\ a_o,\ a_1,\ a_2 \ldots\ldots\ldots a_n\ are\ coefficients$

$This\ is\ the\ \text{exp}ansion\ of:$

$[f(x)]\ about\ (a)$


y
x
a
x-a

$$Determination\ of\ \ Coefficients$$

$$1-a_o \rightarrow by\ \ put\ \ x=a$$

$$f(x)=f(a)=a_o+a_1(a-a)+a_2(a-a)^2.....+a_n(a-a)^n$$

$$\therefore\ a_o=f(a)$$

$$2-a_1 \rightarrow\ \ from\ \frac{df(x)}{dx},\ put\ \ x=a$$

$$\frac{df(x)}{dx}=f'(x)=0+a_1+2a_2(x-a)+........$$

$$f'(a)=0+a_1+0$$

$$\therefore\ a_1=f'(a)$$

$$3-a_2 \rightarrow \ from\ \frac{d^2 f(x)}{dx^2}$$

$$\frac{d^2 f(x)}{dx^2} = f''(x) = 0+0+2a_2+6a_3(x-a)+........$$

$$Put\ \ x=a$$

$$f''(a) = 0+0+2a_2+0$$

$$\therefore\ a_2 = \frac{f''(a)}{2}$$

$$4-a_3 \rightarrow \ from\ \frac{d^3 f(x)}{dx^3}$$

$$So;\ \ a_3 = \frac{f'''(a)}{6} = \frac{f'''(a)}{3!}$$

$$\therefore\ f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+\frac{f'''(a)}{3!}(x-a)^3 + \dots\dots\dots\dots\dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\therefore\ f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\ (x-a)^n$$

If $(a = 0)$ the expansion is about the origin.

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots\dots\dots + f^{(n)}(0)\frac{x^n}{n!}$$

Maclurian Series

Expansion of $[f(x+\Delta x)]$

Write; $x = a;$ $x - a = \Delta x$

$$f(x+\Delta x) = f(x) + f'(x)(\Delta x) + f''(x)\frac{(\Delta x)^2}{2!}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots + f^{(n)}(x)\frac{(\Delta x)^n}{n!}$$

$$Ex. (1)\ Find\ \sqrt{17}$$

Solution

$$x = 16 \quad \Delta x = 17 - 16 = 1$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{4} x^{-\frac{3}{2}}$$

$$\sqrt{17} = \sqrt{16} + \frac{1}{2}(16)^{-\frac{1}{2}}(1) + [-\frac{1}{4}(16)^{-\frac{3}{2}} \frac{(1)^2}{2!}] = 4.12304$$

$Ex.(2)\ \ Find\ \ \ \sin(28)$

$Solution$

$$f(x) = \sin x, \qquad x = \frac{\pi}{6}; \quad \Delta x = -2 * \frac{\pi}{180}$$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$\sin(28) = \sin(\frac{\pi}{6}) + (\cos(\frac{\pi}{6}))(\frac{-2\pi}{180}) + (-\sin(\frac{\pi}{6}))(\frac{(\frac{-2\pi}{180})^2}{2!})$$

$$= 0.46946 \approx 0.47$$

$$Expand\ \ e^x\ \ in\ \ Maclaurian(Taylor)\ Series$$

$$f(x) = e^x$$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + ... + f^{(n)}(0)\frac{x^n}{n!}$$

$$f(x) = e^x \ \ at\ \ x = 0 \Rightarrow f(0) = 1$$

$$f'(x) = f''(x) = f'''(x) = ...f^{(n)}(x) = e^x$$

$$f'(0) = 1 = f''(0) = f'''(0) = ...f^{(n)}(0)$$

$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}$$

Solution of Higher al-gebaric and Transcendental Equations:

$$-\ f(x) = a_o + a_1 x = 0$$

$$a_o + a_1 x = 0$$

$$a_1 x = -a_o$$

$$\therefore x = \frac{-a_o}{a_1} \quad one\ \ root$$

$$-If \quad f(x) = ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$f(x) = x^6 + 3x^4 - 6x + 7 = 0$$

or

$$f(x) = \sin x + x + e^x = 0$$

or

$$f(x) = \cosh x + \tan x + \ln x = 0$$

$Approximation\ Solution\ used\ to\ Solve\ the\ Equations:$

$Newton-Raphson\ Method$

$Suppose\ f(x)=0;\ find\ the\ root\ x,$

$Start\ with\ an\ initial\ estimate\ (x^{(o)})\ and\ improve$

$estimate\ to,\ x^{(1)} = x^{(o)} - \dfrac{f(x^{(o)})}{f'(x^{(o)})}\ after\ repeat\ and$

$obtain\ (x^{(2)}\ and\ x^{(3)})\ and\ after\ (k+1)\ cycles;$

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

$$Derivation\ of\ Newton-Raphson\ Formula:$$

$$By\ Taylor\ Series;$$

$$Let\ (x^{(o)})\ initial\ estimate\ for\ the\ root\ \ [f(x)=0]$$

$$f(x^{(o)}) \neq 0$$

$$x^{(1)} = x^{(o)} + \Delta x^{(o)}$$

$$\Delta x^{(o)} \rightarrow is\ the\ correction$$

$$f(x^{(o)} + \Delta\ x^{(o)}) = f(x^{(o)}) + f'(x^{(o)})(\Delta x^{(o)}) + ...$$

$$0 = f(x^{(o)}) + f'(x^{(o)})(\Delta x^{(o)})$$

$$-f(x^{(o)}) = f'(x^{(o)})(\Delta x^{(o)})$$

$$-f(x^{(o)}) = f'(x^{(o)})(\Delta x^{(o)})$$

$$(\Delta x^{(o)}) = -\frac{f(x^{(o)})}{f'(x^{(o)})}$$

$$x^{(1)} = x^{(o)} + \Delta x^{(o)}$$

$$x^{(1)} = x^{(o)} - \frac{f(x^{(o)})}{f'(x^{(o)})}$$

By repetition:

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

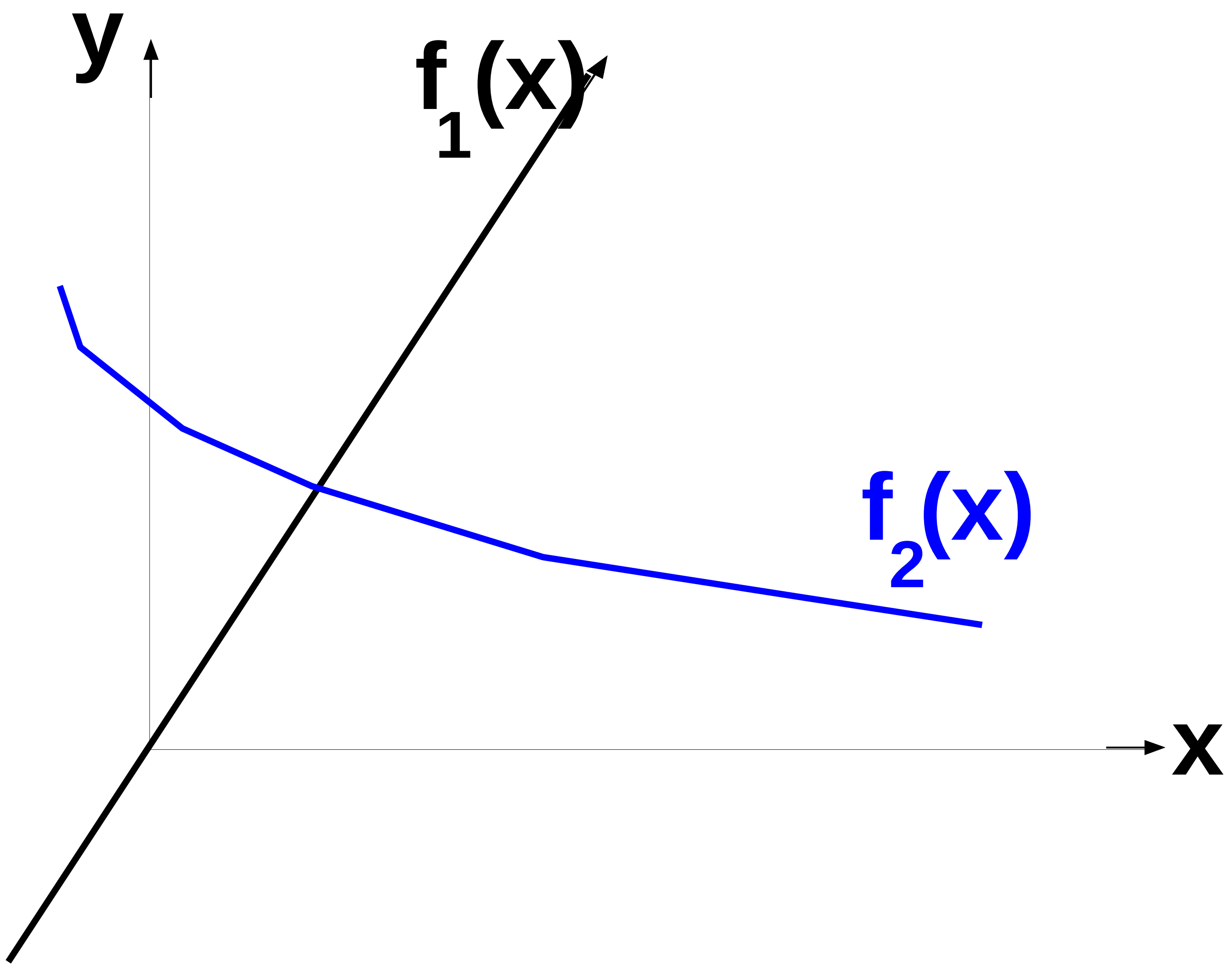
$$Ex1: Solve\ \ 2x - e^{-x} = 0$$

$$Solution:$$

$$2x = e^{-x}$$

$$f_1(x) = 2x \ \ , \quad f_2 = e^{-x}$$

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$


y
f1(x)
f2(x)
x

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

$$f'(x) = 2 + e^{-x}$$

$$Start \ \ x^o = 0$$

$$x^1 = x^o - \frac{2x^o - e^{-x^o}}{2 + e^{-x^o}} = 0 - \frac{2(0) - e^{-0}}{2 + e^{-0}} = \frac{1}{3} = 0.333$$

$$x^2 = 0.333 - \frac{2(0.333) - e^{-0.333}}{2 + e^{-0.333}} = 0.3518$$

$$x^3 = 0.3518 - \frac{2(0.3518) - e^{-0.3518}}{2 + e^{-0.3518}} = 0.3517 \approx 0.3518$$

$$\therefore x = 0.3517$$

$$Ex2: Use\ \ Newton-Raphson\ Method\ \ to\ solve;$$

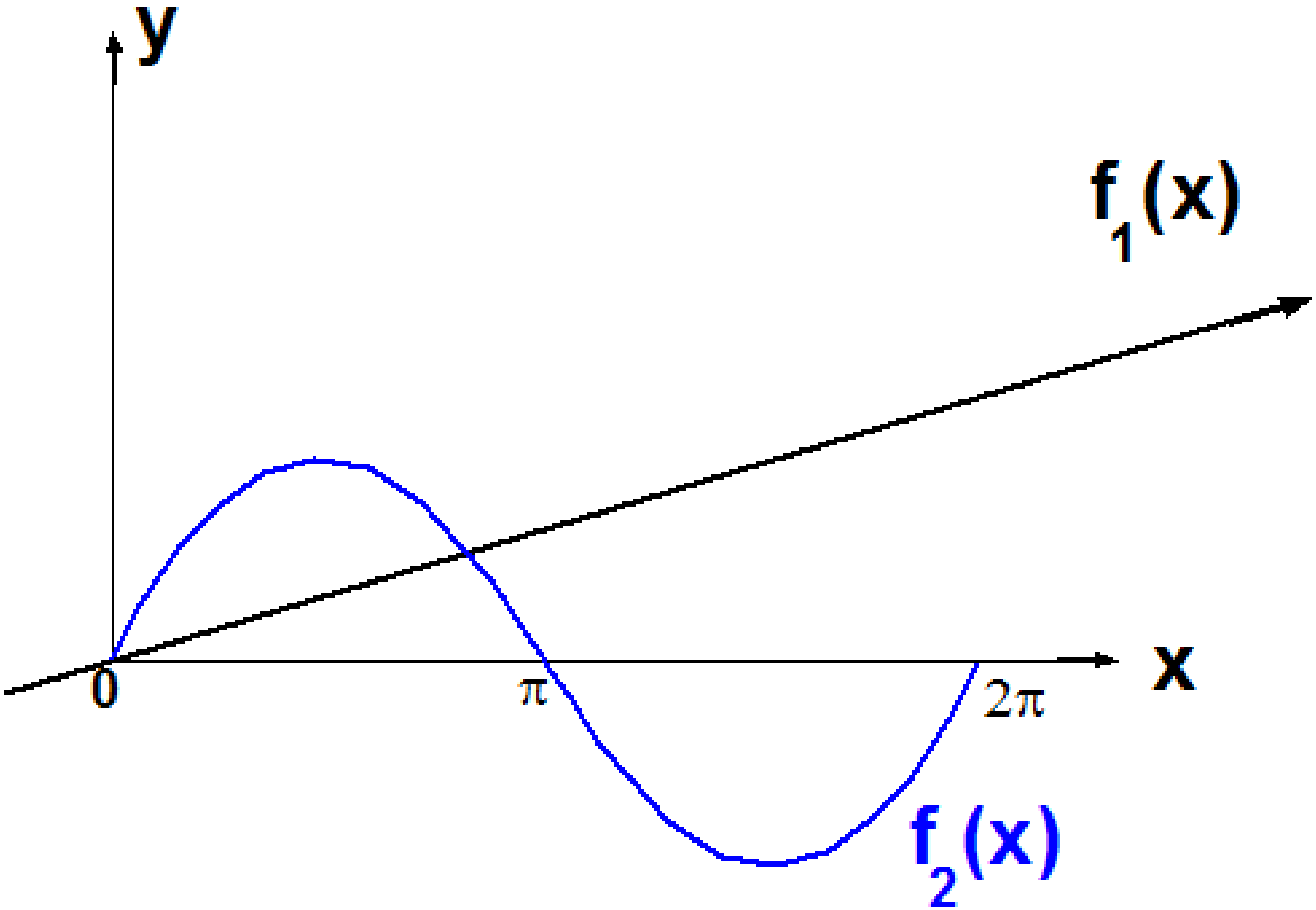
$$0.25x - \sin x = 0\ \ (required\ the\ root > 0)$$

$$Solution:$$

$$f_1(x) = 0.25x\ \ ,\quad f_2 = \sin x$$

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

$$Start\ \ x^o = 3$$


y
f1(x)
0
π
2π
x
f2(x)

$$x^1 = x^o - \frac{f(x^o)}{f'(x^o)} = 3 - \frac{0.25(3) - \sin(3)}{0.25 - \cos(3)}$$

$$= 2.50896 \approx 2.51$$

$$x^2 = x^1 - \frac{f(x^1)}{f'(x^1)} = 2.51 - \frac{0.25(2.51) - \sin(2.51)}{0.25 - \cos(2.51)}$$

$$= 2.475 \approx 2.51$$

$$\therefore x = 2.475$$

Iteration Method

To solve $(f(x)=0)$, transform $(f(x)=0)$ to the form of $(x=g(x))$. If (x^{k}) an estimate for (x) will improve as;

$$x^{k+1} = g(x)^{k}$$

The estimation is convergence when;

$$\left| g'(x)^{k} \right| < 1$$

$$Ex1: Solve\ \ 4x-6=0$$

$$Solution:$$

$$x+3x=6$$

$$x=6-3x$$

$$x=g(x)=6-3x$$

$$g'(x)=-3$$

$$|-3|>1 \quad No \quad convergence$$

$$or: \quad x + 3x = 6$$

$$3x = 6 - x$$

$$x = \frac{1}{3}(6 - x)$$

$$g(x) = \frac{1}{3}(6 - x)$$

$$g'(x) = -\frac{1}{3}$$

$$\left| -\frac{1}{3} \right| = \frac{1}{3} < 1$$

$$\therefore o.k.$$

$$x = \frac{1}{3}(6 - x)$$

$$x^o = 0$$

$$x^1 = \frac{1}{3}(6 - 0) = 2$$

$$x^2 = \frac{1}{3}(6 - 2) = 1.333$$

$$x^3 = \frac{1}{3}(6 - 1.333) = 1.556$$

$$x^4 = \frac{1}{3}(6 - 1.556) = 1.481$$

$$x^5 = \frac{1}{3}(6 - 1.481) = 1.506$$

$$x^6 = \frac{1}{3}(6 - 1.506) = 1.498$$

$$x^7 = \frac{1}{3}(6 - 1.498) = 1.5006$$

$$\therefore x = 1.5006$$

$$Ex1: Solve\ \ x^2 - e^{-x} = 0$$

Solution:

$$x^2 = e^{-x}$$

$$x = e^{-0.5x}$$

$$g(x) = e^{-0.5x}$$

$$g'(x) = -0.5e^{-0.5x}$$

$$x^o = 0$$

$$g'(x) = -0.5e^{-0.5(0)} = -0.5$$

$$\left|-0.5\right| < 1 \quad \therefore o.k. \quad Cognv.$$

$$Put \quad x^{o} = 0$$

$$x^{k+1} = g(x)^{k}$$

$$x^{1} = e^{-0.5(0)} = 1$$

$$x^{2} = e^{-0.5(1)} = 0.6065$$

$$x^{3} = e^{-0.5(0.6065)} = 0.738$$

$$x^{4} = e^{-0.5(0.738)} = 0.691$$

$$x^{5} = e^{-0.5(0.691)} = 0.708$$

$$x^{6} = e^{-0.5(0.708)} = 0.702$$

$$x^{7} = e^{-0.5(0.702)} = 0.704 \approx 0.702$$

$$\therefore x = 0.702$$